

A note on generators of the endomorphism semigroup of an infinite countable chain

Ilinka Dimitrova, Vítor H. Fernandes* and Jörg Koppitz

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Abstract

In this note, we consider the semigroup $\mathcal{O}(X)$ of all order endomorphisms of an infinite chain X and the subset J of $\mathcal{O}(X)$ of all transformations α such that $|\text{Im}(\alpha)| = |X|$. For an infinite countable chain X , we give a necessary and sufficient condition on X for $\mathcal{O}(X) = \langle J \rangle$ to hold. We also present a sufficient condition on X for $\mathcal{O}(X) = \langle J \rangle$ to hold, for an arbitrary infinite chain X .

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Introduction

The *rank* of a semigroup S is the minimum cardinality of a generating set of S . For a countable semigroup S , in particular, for a finitely generated semigroup S , determining the rank of S is a natural question. Contrariwise, for an uncountable semigroup S , this concept has no interest, since the rank of S is always $|S|$. This last fact leads to the following notion. For a subset A of a semigroup S , the *relative rank* of S modulo A is the minimum cardinality of a subset B of S such that $\langle A \cup B \rangle = S$. This cardinal is denoted by $\text{rank}(S : A)$. It follows immediately from the definition that $\text{rank}(S : A) = \text{rank}(S : \langle A \rangle)$ and that $\text{rank}(S : A) = 0$ if and only if A is a generating set of S .

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The notion of relative rank was introduced by Ruškuc in [8], who proved that the rank of a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$, with the sandwich matrix P in normal form, is equal to $\max\{|I|, |\Lambda|, \text{rank}(G : H)\}$, where H is the subgroup of G generated by the entries of P . In [6], Howie et al. considered the relative ranks of the full transformation semigroup $\mathcal{T}(X)$ on X , where X is an infinite set, modulo some distinguished subsets of $\mathcal{T}(X)$. They showed that $\text{rank}(\mathcal{T}(X) : \mathcal{S}(X)) = 2$, $\text{rank}(\mathcal{T}(X) : \mathcal{E}(X)) = 2$ and $\text{rank}(\mathcal{T}(X) : J) = 0$, where $\mathcal{S}(X)$ is the symmetric group on X , $\mathcal{E}(X)$ is the set of all idempotent transformations on X and J is the top \mathcal{J} -class of $\mathcal{T}(X)$, i.e. $J = \{\alpha \in \mathcal{T}(X) \mid |\text{Im}(\alpha)| = |X|\}$.

Throughout this paper, we will represent a chain only by its support set and, as usual, its order by the symbol \leq . Let X be a chain. A transformation α of X is said to be *order-preserving* or an (order) *endomorphism* of X if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X$. We denote by $\mathcal{O}(X)$ the subsemigroup of $\mathcal{T}(X)$ of all (order) endomorphisms of X .

For a finite chain X , it is well known, and clear, that $\mathcal{O}(X)$ is a regular semigroup. The problem for an infinite chain X is much more involved. Nevertheless, more generally, a characterization of those posets P for which the semigroup of all endomorphisms of P is regular was done by Aïzenštat in 1968 [2] and, independently, by Adams and Gould in 1989 [1].

Let X be an infinite chain. A useful regularity criterion for the elements of $\mathcal{O}(X)$ was proved in [7] by Mora and Kemprasit, who deduced several previous known results based on it: for instance, that $\mathcal{O}(\mathbb{Z})$ is regular while $\mathcal{O}(\mathbb{Q})$ and $\mathcal{O}(\mathbb{R})$ are not regular, by considering their usual orders. In [3], Fernandes et al. described the largest regular subsemigroup of $\mathcal{O}(X)$ and also Green's relations on $\mathcal{O}(X)$. The relative rank of $\mathcal{T}(X)$ modulo the subsemigroup $\mathcal{O}(X)$ was considered by Higgins et al. in [4]. They showed that $\text{rank}(\mathcal{T}(X) : \mathcal{O}(X)) = 1$, when X is an arbitrary countable chain or an arbitrary well-ordered set, while $\text{rank}(\mathcal{T}(\mathbb{R}) : \mathcal{O}(\mathbb{R}))$ is uncountable, by considering the usual order of \mathbb{R} .

For a fixed chain X , consider the following two subsets of the semigroup $\mathcal{O}(X)$:

$$J = \{\alpha \in \mathcal{O}(X) \mid |\text{Im}(\alpha)| = |X|\} \quad \text{and} \quad J_f = \{\alpha \in \mathcal{O}(X) \mid |\text{Im}(\alpha)| < \aleph_0\}.$$

Notice that J_f is clearly an ideal of $\mathcal{O}(X)$. On the other hand, unlike the analogous set for $\mathcal{T}(X)$, J is not necessarily a \mathcal{J} -class of $\mathcal{O}(X)$ (see [3]).

In this note we study the relative rank of the semigroup $\mathcal{O}(X)$ modulo J . For an infinite countable chain X , we give a necessary and sufficient condition on X for $\mathcal{O}(X) = \langle J \rangle$ to hold (notice that, for a finite X , $\mathcal{O}(X) =$

$\langle J \rangle$ if and only if $|X| = 1$). We also present a sufficient condition on X for $\mathcal{O}(X) = \langle J \rangle$ to hold, for an arbitrary infinite chain X .

For general background on Semigroup Theory, we refer the reader to Howie's book [5].

1 Main results

Let X be an infinite chain. Let $x \in X$ and define

$$[x] = \{y \in X \mid y \leq x\} \quad \text{and} \quad]x] = \{y \in X \mid x \leq y\}$$

(i.e the left and right order ideals generated by x). Define also

$$X^0 = \{x \in X \mid |[x]| = |X| = |[x]| \},$$

$$X^- = \{x \in X \mid |[x]| < |X|\}$$

and

$$X^+ = \{x \in X \mid |[x]| < |X|\}.$$

Notice that, since X is an infinite set, if $x \in X^-$ (respectively, $x \in X^+$) then $|X| = |[x]|$ (respectively, $|X| = |[x]|$). Hence X is a disjoint union of X^- , X^0 and X^+ .

Let us consider the sets \mathbb{N} , $\mathbb{Z}_- = \mathbb{Z} \setminus (\mathbb{N} \cup \{0\})$, \mathbb{Z} , \mathbb{Q} and \mathbb{R} , with their usual orders. Then, we have:

1. $X^- = \mathbb{N}$, $X^0 = \emptyset$ and $X^+ = \emptyset$, if $X = \mathbb{N}$;
2. $X^- = \emptyset$, $X^0 = \emptyset$ and $X^+ = \mathbb{Z}_-$, if $X = \mathbb{Z}_-$;
3. $X^- = \emptyset$, $X^0 = X$ and $X^+ = \emptyset$, for $X \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

Recall that, given two posets P and Q with disjoint supports, the *ordinal sum* $P \oplus Q$ of P and Q (by this order) is the poset with support $P \cup Q$ such that P and Q are subposets of $P \oplus Q$ and $x < y$, for all $x \in P$ and $y \in Q$. This operation on posets is associative (but not commutative). For our purposes, it is convenient to admit empty posets.

Let \mathbb{Z} be the chain $\mathbb{N} \oplus \{0\} \oplus \mathbb{Z}_-$, with the usual orders on \mathbb{N} and \mathbb{Z}_- . Then, being $X = \mathbb{Z}$, we have $X^- = \mathbb{N}$, $X^0 = \{0\}$ and $X^+ = \mathbb{Z}_-$.

By considering X^- , X^0 and X^+ as subposets of X , we have the following decomposition of X :

Lemma 1.1. *Let X be an infinite chain. Then $X = X^- \oplus X^0 \oplus X^+$.*

Proof. First, let $a \in X^-$ and $b \in X^0 \cup X^+$. If $b \leq a$ then $(b) \subseteq (a)$ and so $|X| = |(b)| \leq |(a)| < |X|$, a contradiction. Then $a < b$. On the other hand, given $a \in X^- \cup X^0$ and $b \in X^+$, by a dual reasoning, we may show that $a < b$. This proves the lemma. \square

Note 1.2. *Let X be an infinite chain and let $\alpha \in \mathcal{O}(X)$. If there exist $x^+ \in X^+$ and $x^- \in X^-$ such that $x^+\alpha = x^-$ or $x^-\alpha = x^+$ then $\alpha \notin J$.*

In fact, suppose that $x^+\alpha = x^-$ (the other case can be treated dually). Then $\text{Im}(\alpha) \subseteq (x^-) \cup [x^+)\alpha$ and so $|\text{Im}(\alpha)| \leq |(x^-)| + |[x^+)\alpha| \leq |(x^-)| + |[x^+)| < |X| + |X| = |X|$, i.e. $\alpha \notin J$, as required.

Note 1.3. *Let X be an infinite chain such that $X^0 = \emptyset$. Let $\alpha \in \mathcal{O}(X)$ be such that $x^+\alpha = x^-$ or $x^-\alpha = x^+$, for some $x^+ \in X^+$ and $x^- \in X^-$. Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{O}(X)$ be such that $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$. Then $|\text{Im}(\alpha)| \leq |\text{Im}(\alpha_i)| < |X|$ (and so $\alpha, \alpha_i \notin J$), for some $i = 1, 2, \dots, n$.*

In fact, for the case $x^+\alpha = x^-$ (the other case is dual), let $i = \min\{j \in \{1, \dots, n\} \mid x^+\alpha_1 \cdots \alpha_j \in X^-\}$. Then $x^+\alpha_1 \cdots \alpha_{i-1} \in X^+$ (for $i = 1$ the expression $x^+\alpha_1 \cdots \alpha_{i-1}$ has the meaning of x^+) and $(x^+\alpha_1 \cdots \alpha_{i-1})\alpha_i \in X^-$, since $X^0 = \emptyset$. Hence, by Note 1.2, $\alpha_i \notin J$. On the other hand, from the equality $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, it follows that $|\text{Im}(\alpha)| \leq |\text{Im}(\alpha_j)|$ (indeed $\text{Im}(\alpha_1 \cdots \alpha_j) \subseteq \text{Im}(\alpha_j)$, whence $\text{Im}(\alpha) = (\text{Im}(\alpha_1 \cdots \alpha_j))(\alpha_{j+1} \cdots \alpha_n) \subseteq (\text{Im}(\alpha_j))(\alpha_{j+1} \cdots \alpha_n)$ and so $|\text{Im}(\alpha)| \leq |(\text{Im}(\alpha_j))(\alpha_{j+1} \cdots \alpha_n)| \leq |\text{Im}(\alpha_j)|$), for all $j \in \{1, \dots, n\}$, and so $|\text{Im}(\alpha)| \leq |\text{Im}(\alpha_i)| < |X|$, as required.

This last note can be rewritten as follows:

Lemma 1.4. *Let X be an infinite chain such that $X^0 = \emptyset$ and let $\alpha \in \mathcal{O}(X)$ be such that $X^+\alpha \cap X^- \neq \emptyset$ or $X^-\alpha \cap X^+ \neq \emptyset$. Then $\alpha \notin \langle J \rangle$.*

Before presenting our next note, we introduce the following (natural) notation. For $x \in X$ and $Y \subseteq X$, by $x < Y$ (respectively, $x > Y$) we mean that $x < y$ (respectively, $x > y$), for all $y \in Y$.

Note 1.5. Let X be an infinite chain and let $\alpha \in \mathcal{O}(X)$.

1. *If $b \in \text{Im}(\alpha)$ and there exists no element $c \in X$ such that $c < b\alpha^{-1}$ then $\text{Im}(\alpha) \subseteq [b)$.*

In fact, let $y \in \text{Im}(\alpha)$. Take $x \in y\alpha^{-1}$. Then $x \not< b\alpha^{-1}$ and so there exists $a \in b\alpha^{-1}$ such that $a \leq x$. It follows that $b = a\alpha \leq x\alpha = y$, whence $y \in [b)$, as required.

2. If $\alpha \in J$ and $b \in \text{Im}(\alpha) \cap X^+$ then there exists an element $c \in X$ such that $c < b\alpha^{-1}$.

In fact, if there exists no element $c \in X$ such that $c < b\alpha^{-1}$ then, by 1 above, we have $\text{Im}(\alpha) \subseteq [b]$ and, as $b \in X^+$, it follows $|\text{Im}(\alpha)| \leq |[b]| < |X|$, whence $\alpha \notin J$, a contradiction.

3. If $\alpha \in J$ and $y \in X^+$ then there exists an element $b \in \text{Im}(\alpha)$ such that $b < y$.

In fact, if $y \leq b$, for all $b \in \text{Im}(\alpha)$, then $\text{Im}(\alpha) \subseteq [y]$ and, as $y \in X^+$, it follows $|\text{Im}(\alpha)| \leq |[y]| < |X|$, whence $\alpha \notin J$, a contradiction.

By combining 2 and 3 of the previous note, it follows immediately:

Note 1.6. Let X be an infinite chain such that $X = X^+$, let $\alpha, \beta \in J$ and let $b \in \text{Im}(\alpha)$. Then there exist $c \in X$ and $b' \in \text{Im}(\beta)$ such that $b' < c < b\alpha^{-1}$.

From 3 of Note 1.5, if $X = X^+$, it is clear that $\text{Im}(\alpha)$ has no lower bounds, for all $\alpha \in J$. Moreover, we have:

Lemma 1.7. Let X be an infinite chain such that $X = X^+$ (respectively, $X = X^-$) and let $\alpha \in \langle J \rangle$. Then $\text{Im}(\alpha)$ has no minimum (respectively, maximum). In particular $J_f \cap \langle J \rangle = \emptyset$.

Proof. We prove this result for $X = X^+$. The case $X = X^-$ is dual.

By contradiction, let us suppose that $\text{Im}(\alpha)$ has minimum. Denote $\min \text{Im}(\alpha)$ by b_n .

As $\alpha \in \langle J \rangle$, we have $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$, for some $\alpha_1, \alpha_2, \dots, \alpha_n \in J$.

Notice that, since $b_n \in \text{Im}(\alpha)$, we also have $b_n \in \text{Im}(\alpha_n)$. By applying Note 1.6, we find elements $c_n \in X$ and $b_{n-1} \in \text{Im}(\alpha_{n-1})$ such that

$$b_{n-1} < c_n < b_n \alpha_n^{-1}.$$

By applying again Note 1.6, we can take elements $c_{n-1} \in X$ and $b_{n-2} \in \text{Im}(\alpha_{n-2})$ such that

$$b_{n-2} < c_{n-1} < b_{n-1} \alpha_{n-1}^{-1}.$$

Moreover, by Note 1.6, we may recursively construct two sequences

$$c_n, c_{n-1}, \dots, c_2 \quad \text{and} \quad b_{n-1}, b_{n-2}, \dots, b_1$$

of elements of X such that $b_{i-1} \in \text{Im}(\alpha_{i-1})$ and

$$b_{i-1} < c_i < b_i \alpha_i^{-1},$$

for $i = 2, \dots, n$. In addition, by Note 1.5, we may also consider an element $c_1 \in X$ such that $c_1 < b_1\alpha_1^{-1}$.

Let $i = 1, 2, \dots, n$. Then, $c_i\alpha_i < b_i$. In fact, since $c_i < b_i\alpha_i^{-1}$, we get $c_i \notin b_i\alpha_i^{-1}$, whence $c_i\alpha_i \neq b_i$, and, given $a \in b_i\alpha_i^{-1}$, we have $c_i < a$ and so $c_i\alpha_i \leq a\alpha_i = b_i$.

Next, by induction on i , we prove that $c_1\alpha_1\alpha_2 \cdots \alpha_i < b_i$, for $i = 1, 2, \dots, n$. Let $i = 1$. Then, the inequality $c_1\alpha_1 < b_1$ was already proved above. Hence, let $i > 1$ and suppose that $c_1\alpha_1\alpha_2 \cdots \alpha_{i-1} < b_{i-1}$, by induction hypothesis. Since $b_{i-1} < c_i$, we have $c_1\alpha_1\alpha_2 \cdots \alpha_{i-1} < c_i$ and so $c_1\alpha_1\alpha_2 \cdots \alpha_{i-1}\alpha_i \leq c_i\alpha_i < b_i$, as required.

Hence, in particular, we have $c_1\alpha = c_1\alpha_1\alpha_2 \cdots \alpha_n < b_n = \min \text{Im}(\alpha)$, which is a contradiction. Therefore, $\text{Im}(\alpha)$ has no minimum, as required. \square

Next, we state our fundamental lemma.

Main Lemma. *Let X be an infinite chain. Then $J_f \subseteq \langle J \rangle$ if and only if $X^0 \neq \emptyset$.*

Proof. First, suppose that $X^0 = \emptyset$. If $X^+ = \emptyset$ or $X^- = \emptyset$ then, by Lemma 1.7, we have $J_f \cap \langle J \rangle = \emptyset$, whence $J_f \not\subseteq \langle J \rangle$ (notice that $J_f \neq \emptyset$). On the other hand, admit that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Fix $a \in X^-$ and let $\alpha \in \mathcal{O}(X)$ be the constant transformation with image $\{a\}$. Then $X^+\alpha \cap X^- \neq \emptyset$ and so, by Lemma 1.4, $\alpha \notin \langle J \rangle$. Since $\alpha \in J_f$, in this case, we also obtain $J_f \not\subseteq \langle J \rangle$.

Conversely, suppose that $X^0 \neq \emptyset$ and fix an element $0 \in X^0$. Let $\alpha \in J_f$.

Suppose, without loss of generality, that $0\alpha \leq 0$ (the case $0\alpha \geq 0$ can be treated dually).

We begin by defining a transformation $\beta \in \mathcal{O}(X)$ by

$$x\beta = \begin{cases} x\alpha, & x \leq 0 \\ x, & x > 0. \end{cases}$$

Next, let $\text{Im}(\alpha) = \{a_1 < a_2 < \cdots < a_n\}$, $n \in \mathbb{N}$, and suppose that $0\alpha = a_i < a_{i+1} < \cdots < a_{i+k} \leq 0$, with $a_{i+k+1} > 0$ or $i+k = n$, for some $i \in \{1, \dots, n\}$ and a non-negative integer k . Then, for $0 \leq j \leq k$, we define transformations $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$ of $\mathcal{O}(X)$ by

$$x\gamma_1^{(j)} = \begin{cases} x, & x < 0 \\ 0, & x \geq 0 \text{ and } x \not\geq a_{i+j}\alpha^{-1} \\ x, & x > a_{i+j}\alpha^{-1} \end{cases} \text{ and } x\gamma_2^{(j)} = \begin{cases} x, & x < a_{i+j} \\ a_{i+j}, & a_{i+j} \leq x \leq 0 \\ x, & x > 0. \end{cases}$$

Finally, we define a transformation $\delta \in \mathcal{O}(X)$ as being the identity map on X if $i + k = n$ and by

$$x\delta = \begin{cases} x, & x \leq 0 \\ a_{i+k+1}, & x > 0 \text{ and } x \not\geq a_{i+k+1}\alpha^{-1} \\ x\alpha, & x > a_{i+k+1}\alpha^{-1} \end{cases}$$

otherwise.

Since $0 \in X^0$, it is clear that $\beta, \gamma_1^{(0)}, \gamma_2^{(0)}, \dots, \gamma_1^{(k)}, \gamma_2^{(k)}, \delta \in J$. Moreover, we have $\alpha = \beta\gamma_1^{(0)}\gamma_2^{(0)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta$. In fact, taking $x \in X$, we may consider three cases:

1. If $x \leq 0$ then

$$(x)\beta\gamma_1^{(0)}\gamma_2^{(0)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta = (x\alpha)\gamma_1^{(0)}\gamma_2^{(0)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta = x\alpha,$$

since $x\alpha \leq a_i$;

2. If $x > 0$ and $x\alpha \leq 0$ then there exists $j \in \{0, 1, \dots, k\}$ such that $x\alpha = a_{i+j}$ and

$$\begin{aligned} (x)\beta\gamma_1^{(0)}\gamma_2^{(0)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta &= (x)\gamma_1^{(j)}\gamma_2^{(j)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta \\ &= (0)\gamma_2^{(j)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta \\ &= (a_{i+j})\gamma_1^{(j+1)}\gamma_2^{(j+1)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta \\ &= a_{i+j} \\ &= x\alpha; \end{aligned}$$

3. If $x > 0$ and $x\alpha > 0$ then

$$(x)\beta\gamma_1^{(0)}\gamma_2^{(0)} \dots \gamma_1^{(k)}\gamma_2^{(k)}\delta = x\delta = x\alpha.$$

Thus $\alpha \in \langle J \rangle$ and so $J_f \subseteq \langle J \rangle$, as required. \square

The following observation will be useful in the proof of our next result.

Note 1.8. Let X be an infinite chain. Then J_f contains elements of arbitrary finite (non null) rank. In fact, for all $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X$, with $x_1 < x_2 < \dots < x_n$, we may construct transformations $\alpha \in \mathcal{O}(X)$ such that $\text{Im}(\alpha) = \{x_1, x_2, \dots, x_n\}$. For instance, the transformation α on X defined by

$$x\alpha = \begin{cases} x_1, & x \leq x_1 \\ x_i, & x_{i-1} < x \leq x_i \text{ and } 2 \leq i \leq n-1 \\ x_n, & x_{n-1} < x \end{cases}$$

belongs to J_f .

Notice that if X is an infinite countable chain then $J_f = \mathcal{O}(X) \setminus J$. Thus, in this case, by the previous lemma, we obtain that $\mathcal{O}(X) = \langle J \rangle$ if and only if $X^0 \neq \emptyset$. Furthermore, we have:

Theorem 1.9. *Let X be an infinite countable chain. The following properties are equivalent:*

1. $\mathcal{O}(X) = \langle J \rangle$, i.e. $\text{rank}(\mathcal{O}(X) : J) = 0$;
2. $\text{rank}(\mathcal{O}(X) : J) < \aleph_0$;
3. $X^0 \neq \emptyset$.

Proof. Notice that 1 trivially implies 2 and, by the previous lemma, 3 implies 1, whence it remains to prove that 2 implies 3. Thus, suppose that $X^0 = \emptyset$. Let \mathcal{C} be a generating set of $\mathcal{O}(X)$.

First, we admit that $X^- \neq \emptyset$ and $X^+ \neq \emptyset$. As $X^0 = \emptyset$, we must have $|X^-| = \aleph_0$ or $|X^+| = \aleph_0$. Suppose, without loss of generality, that $|X^-| = \aleph_0$ (the case $|X^+| = \aleph_0$ can be treated dually). Hence, given $n \in \mathbb{N}$, we may consider n elements $x_1, x_2, \dots, x_n \in X^-$, with $x_1 < x_2 < \dots < x_n$, and the transformation $\alpha \in \mathcal{O}(X)$ such that $\text{Im}(\alpha) = \{x_1, x_2, \dots, x_n\}$ constructed in Note 1.8. Then, for any $x^+ \in X^+$, we have $x^+ \alpha = x_n \in X^-$ and so, accordingly with Note 1.3, \mathcal{C} contains a transformation $\beta \in \mathcal{O}(X)$ such that $n = |\text{Im}(\alpha)| \leq |\text{Im}(\beta)| < \aleph_0$. Thus, as $n \in \mathbb{N}$ is arbitrary, \mathcal{C} must contain an infinite number of elements of J_f .

On the other hand, admit that $X^+ = \emptyset$ or $X^- = \emptyset$. Then, by Lemma 1.7, we have $J_f \cap \langle J \rangle = \emptyset$. Let $n \in \mathbb{N}$, let $x_1, x_2, \dots, x_n \in X$ be such $x_1 < x_2 < \dots < x_n$ and consider the transformation $\alpha \in \mathcal{O}(X)$ such that $\text{Im}(\alpha) = \{x_1, x_2, \dots, x_n\}$ constructed in Note 1.8. Let $\alpha_1, \dots, \alpha_k \in \mathcal{C}$ ($k \in \mathbb{N}$) be such that $\alpha = \alpha_1 \cdots \alpha_k$. Since $J_f \cap \langle J \rangle = \emptyset$ and $\alpha \in J_f$, we must have $\alpha_i \notin J$, for some $i \in \{1, \dots, k\}$. Hence, $n = |\text{Im}(\alpha)| \leq |\text{Im}(\alpha_i)| < \aleph_0$ (check the proof of Note 1.3). Thus, as $n \in \mathbb{N}$ is arbitrary, also in this case, \mathcal{C} must contain an infinite number of elements of J_f .

Therefore, $\text{rank}(\mathcal{O}(X) : J) \geq \aleph_0$, as required. \square

Recall that, for $X \in \{\mathbb{Z}, \mathbb{Q}\}$, with the usual order, we have $X^0 = X$. Therefore, as an immediate consequence of the last theorem, we obtain:

Corollary 1.10. *Let $X \in \{\mathbb{Z}, \mathbb{Q}\}$, with the usual order. Then $\mathcal{O}(X) = \langle J \rangle$.*

Notice that, for the chain $X = \mathbb{Z}$ defined in the beginning of this section, we have $X^0 = \{0\}$, whence also in this case $\mathcal{O}(X) = \langle J \rangle$.

On the contrary, we have:

Proposition 1.11. *With the usual order of \mathbb{N} , we have $\text{rank}(\mathcal{O}(\mathbb{N}) : J) = \aleph_0$.*

Proof. We already observed that $X^0 = \emptyset$, for $X = \mathbb{N}$ equipped with the usual order. Hence, by Theorem 1.9, we obtain $\text{rank}(\mathcal{O}(\mathbb{N}) : J) \geq \aleph_0$. On the other hand, since $J_f = \mathcal{O}(\mathbb{N}) \setminus J$, we have $\text{rank}(\mathcal{O}(\mathbb{N}) : J) \leq |J_f|$. Therefore, this result follows by showing that $|J_f| = \aleph_0$. In fact, for each $n \in \mathbb{N}$ and each fixed subset $\{x_1, \dots, x_n\}$ of \mathbb{N} with n elements, we have a bijection between the set $\{\alpha \in \mathcal{O}(\mathbb{N}) \mid \text{Im}(\alpha) = \{x_1, \dots, x_n\}\}$ and the set $\mathcal{P}_{n-1}(\mathbb{N} \setminus \{1\})$ of all subsets of $\mathbb{N} \setminus \{1\}$ with $n - 1$ elements, namely $\alpha \mapsto \{\min x_i \alpha^{-1} \mid i = 2, \dots, n\}$. Thus, since the set $\mathcal{P}_f(\mathbb{N})$ of all finite subsets of \mathbb{N} has cardinal \aleph_0 , J_f is an infinite countable union of infinite countable sets and so $|J_f| = \aleph_0$, as required. \square

Observe that our Main Lemma gives us a necessary condition for having $\mathcal{O}(X) = \langle J \rangle$, namely $X^0 \neq \emptyset$. We finish this note by presenting a sufficient condition:

Theorem 1.12. *Let X be an infinite chain such that $X \setminus X^0$ is finite. Then $\mathcal{O}(X) = \langle J \rangle$.*

Proof. Notice that, X^- and X^+ are both finite sets and $|X^0| = |X|$.

Take $\alpha \in \mathcal{O}(X)$.

First, suppose that $X^0 \alpha \cap X^0 \neq \emptyset$. Fix $u, v \in X^0$ such that $u\alpha = v$. If $u \leq v$, we define transformations α_1 and α_2 of $\mathcal{O}(X)$ by

$$x\alpha_1 = \begin{cases} x, & x < u \\ x\alpha, & u \leq x \end{cases} \quad \text{and} \quad x\alpha_2 = \begin{cases} x\alpha, & x < u \\ v, & u \leq x < v \\ x, & v \leq x. \end{cases}$$

On the other hand, if $v < u$, we define transformations α_1 and α_2 of $\mathcal{O}(X)$ by

$$x\alpha_1 = \begin{cases} x\alpha, & x \leq u \\ x, & u < x \end{cases} \quad \text{and} \quad x\alpha_2 = \begin{cases} x, & x < v \\ v, & v \leq x < u \\ x\alpha, & u \leq x. \end{cases}$$

It is a routine matter to show that both cases satisfy $\alpha_1, \alpha_2 \in J$ and $\alpha = \alpha_1 \alpha_2$. Then $\alpha \in \langle J \rangle$.

On the other hand, suppose that $X^0 \alpha \cap X^0 = \emptyset$. Then $\alpha \in J_f$ and so, by our Main Lemma, we obtain again $\alpha \in \langle J \rangle$, as required. \square

Clearly, the converse of this property is not valid in general, as the example $X = \mathbb{Z}$ shows. Nevertheless, as an immediate application, for the usual chain of real numbers, we have:

Corollary 1.13. *With the usual order of \mathbb{R} , we have $\mathcal{O}(\mathbb{R}) = \langle J \rangle$.*

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ILINKA DIMITROVA, Faculty of Mathematics and Natural Science, South-West University "Neofit Rilski", 2700 Blagoevgrad, Bulgaria; email: ilinka_dimitrova@swu.bg

VÍTOR H. FERNANDES, CMA, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal; e-mail: vhf@fct.unl.pt

JÖRG KOPPITZ, Institute of Mathematics, University of Potsdam, 14469 Potsdam, Germany; email: koppitz@uni-potsdam.de